Optimum Time-Frequency Distribution for Detecting a Discrete-Time Chirp Signal in White Gaussian Noise

A. Yasotharan
1103-1760 Main Street West
Hamilton, Ontario, L8S 1H2
Canada
yaso-yasotharan@hotmail.com

T. Thayaparan
Defence R & D Canada - Ottawa
Department of National Defence
3701 Carling Avenue
Ottawa, Ontario, K1A 0Z4
Canada
Thayananthan.Thayaparan@drdc-rddc.gc.ca

Abstract

In the continuous-time domain, Maximum-Likelihood (ML) detection of a chirp signal in white Gaussian noise can be done via the line-integral transform of the classical Wigner distribution. The line-integral transform is known variously as the Hough transform and the Radon transform. For discrete-time signals, the Wigner-type distribution defined by Claasen and Mecklenbrauker has become popular as a signal analysis tool. Moreover, it is commonly believed that ML detection of a discrete-time chirp signal in white Gaussian noise can be done via the line-integral transform of the Wigner-Claasen-Mecklenbrauker distribution. This belief is false and results in loss of performance. We derive a Wigner-type distribution for discrete-time signals whose line-integral transform can be used for ML detection of discrete-time chirp signals in white Gaussian noise. We provide simulated Receiver Operating Curves for the Wigner-Claasen-Mecklenbrauker distribution based method and the new ML-equivalent method and demonstrate the suboptimality of the former.

I. Introduction

For a continuous-time signal $r(t)$, the classical Wigner distribution is defined as [1]

$$W_r(t,\omega) = \int r(t+\tau/2)r^*(t-\tau/2)e^{-j\omega\tau}d\tau, \tag{1}$$

where t is time and ω is frequency. In [1], the Wigner distribution was shown to have many properties that make it a useful signal analysis tool.

Suppose we have observed a continuous-time signal $r(t)$ and want to detect the presence or absence in $r(t)$ of a chirp signal

$$s(t) = ae^{j(\omega_0 t + \frac{1}{2}mt^2)}, \tag{2}$$

with unknown parameters a, ω_0 and m, and with the background being additive white Gaussian noise. The classical maximum-likelihood method is equivalent to the hypothesis test

$$\max_{\omega_0, m} \left| \int r(t)e^{-j(\omega_0 t + \frac{1}{2}mt^2)}dt \right|^2 \underset{\mathbf{H}_0}{\overset{\mathbf{H}_1}{\gtrless}} \gamma, \tag{3}$$

where

- $\mathbf{H}_0$ is the Noise-Only Hypothesis $r(t) = w(t)$,

Paper presented at the RTO SET Symposium on "Target Identification and Recognition Using RF Systems", held in Oslo, Norway, 11-13 October 2004, and published in RTO-MP-SET-080.

- $\mathbf{H}_1$ is the Signal-plus-Noise Hypothesis $r(t) = s(t) + w(t)$,

$w(t)$ is white Gaussian noise and γ is a threshold whose value is set based on probability of error considerations. More precisely, if the maximum on the left hand side (LHS) of (3) is less than the threshold γ then $\mathbf{H}_0$ is considered true and if the maximum on the LHS of (3) is greater than the threshold γ then $\mathbf{H}_1$ is considered true.

In [2], the hypothesis test (3) was shown to be approximately equivalent to

$$\max_{\omega_0, m} \int W_r(t, \omega_0 + mt) \underset{\mathbf{H}_0}{\overset{\mathbf{H}_1}{\gtrless}} \gamma \tag{4}$$

for chirp signals of large duration. This equivalence was shown in [3] to be exact and valid even for finite-duration signals. More specifically, it was shown in [3] that

$$\left| \int r(t) e^{-j(\omega_0 t + \frac{1}{2} m t^2)} dt \right|^2 = \int W_r(t, \omega_0 + mt) dt, \tag{5}$$

where the quantity on the right hand side is a line-integral transform of the Wigner-distribution variously known as the Hough transform and the Radon transform. Detecting a chirp signal via the hypothesis test (3) is known variously as the correlator method and the dechirp-Doppler method.

A study of the use of time-frequency distributions for detecting signals is found in [4].

For a discrete-time signal $r(n)$, the Wigner distribution defined by Claasen and Mecklenbrauker [5] has become popular as a signal analysis tool. Their definition of Wigner distribution is

$$W_r^{CM}(n, \theta) = 2 \sum_k r(n+k) r^*(n-k) e^{-j2k\theta}, \tag{6}$$

where n is discrete-time and θ is frequency.

Suppose we have observed a discrete-time signal $r(n)$, for $n = 0, \ldots, (N-1)$, and want to detect the presence or absence in $r(n)$ of a chirp signal

$$s(n) = \begin{cases} b_0 e^{j(b_1 n + \frac{1}{2} b_2 n^2)} & \text{if } 0 \leq n \leq (N-1), \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

with unknown parameters b_0, b_1 and b_2, with the background being additive white Gaussian noise. Discrete-time chirp signals arise directly in pulse Doppler radars when a target is moving with acceleration [8]. Discrete-time chirp signals also arise in synthetic aperture radars and inverse synthetic aperture radars. A discrete-time chirp signal may also arise as a sampled-version of a continuous-time chirp signal. This is the case, for example, in electronic counter measures to LFM radar and sonar. Many situations where chirp signals occur in nature are described in [9].

For the above discrete-time detection problem, define

$$\Delta_r(c_1, c_2) = \left| \sum_{n=0}^{N-1} r(n) e^{-j(c_1 n + \frac{1}{2} c_2 n^2)} \right|^2. \tag{8}$$

The classical Maximum-Likelihood (ML) method is then equivalent to the hypothesis test

$$\max_{c_1, c_2} \Delta_r(c_1, c_2) \underset{\mathbf{H}_0}{\overset{\mathbf{H}_1}{\gtrless}} \gamma, \tag{9}$$

where

- $\mathbf{H}_0$ is the Noise-Only Hypothesis $r(n) = v(n)$,
- $\mathbf{H}_1$ is the Signal-plus-Noise Hypothesis $r(n) = s(n) + v(n)$,

$v(n)$ is white Gaussian noise and γ is a threshold. We shall henceforth refer to the method of (9) as the *correlator method*. Appendix I gives a brief derivation of the method of (9).

It is commonly and erroneously assumed that the equivalence of (5) for continuous-time signals carries over to discrete-time signals as

$$\left|\sum_n r(n)e^{-j(c_1 n+\frac{1}{2}c_2 n^2)}\right|^2 = \sum_n W_r^{CM}(n, c_1 + c_2 n), \tag{10}$$

where the quantity on the right hand side is the line-integral transform of the Wigner distribution (6). Based on this assumption, it is claimed, erroneously, that ML detection of the discrete-time chirp signal $s(n)$ is equivalent to the hypothesis test

$$\max_{c_1, c_2} \sum_n W_r^{CM}(n, c_1 + c_2 n) \underset{\mathbf{H}_0}{\overset{\mathbf{H}_1}{\gtrless}} \gamma. \tag{11}$$

However, it has been observed in [6] that the W_r^{CM}-based method (11) incurs a 3 dB loss due to non-linearity.[1] In Appendix IV, we provide simulated Receiver Operating Curves for the ML method (9) and the W_r^{CM}-based method (11) and demonstrate the suboptimality of the latter. Moreover, as we will show, the range of unambiguosly measurable values of b_1 for the W_r^{CM}-based method (11) is half of that of the correlator method (9).

In this paper, we derive a time-frequency distribution which is optimum for detecting discrete-time chirp signals in white Gaussian noise, with the optimality being in the sense that ML detection can be carried out via the line-integral transform of the derived time-frequency distribution. The derived time-frequency distribution may be considered a Wigner-type distribution.

It turns out that the Wigner-type time-frequency distribution derived in this paper is the same as that derived by Chan [7] in an effort to solve the problem of aliasing in the Wigner distribution (6). Nevertheless, the optimality property of this distribution for detection of a discrete-time signal was not observed in [7]. Therefore, in the context of signal detection, this discrete-time distribution seems new.

II. Three Wigner-type Time-Frequency Distributions for Discrete-Time Signals

In attempting to write $\Delta_r(c_1, c_2)$ of (8) as the line-integral of a time-frequency distribution of $r(n)$, we arrive at three Wigner-type time-frequency distributions of a discrete-time signal. We first describe these time-frequency distributions and in the next section we describe the actual line-integral transform.

Given a discrete-time signal $r(n)$, we denote $r_\pi(n) = r(n)e^{j\pi n}$. That is $r_\pi(n)$ is the signal obtained by frequency-shifting $r(n)$ by π radians/second.

A. Type-I Wigner Distribution

The type-I Wigner distribution $W_r^I(n, \theta)$ is defined as

$$W_r^I(n,\theta) = \sum_k r(n+k)r^*(n-k)e^{-j2k\theta}, \tag{12}$$

[1] The ratio between the output SNR and the input SNR is (c.f. equation (13) of [6]) $\frac{\text{SNR}_{\text{out}}}{\text{SNR}_{\text{in}}} = \left(\frac{N}{2}\right)\left(\frac{N\text{SNR}_{\text{in}}}{N\text{SNR}_{\text{in}}+1}\right)$, which is less than $N/2$. The ratio approaches $N/2$ as $N\text{SNR}_{\text{in}} \to \infty$.

where n is discrete-time and θ is frequency.

Note that $W_r^I(n,\theta)$ is the same as $W_r^{CM}(n,\theta)$ (c.f. (6)) defined in [5] except for the missing scaling factor 2 at the front.

For a signal $r(n)$ that is zero outside $0 \le n \le (N-1)$, the type-I Wigner distribution $W_r^I(n,\theta)$ is zero outside $0 \le n \le (N-1)$.

The following properties are easy to verify: $W_r^I(n,\theta)$ is real, $W_r^I(n,\theta)$ is a periodic function of θ with period π, $W_{r_\pi}^I(n,\theta) = W_r^I(n,\theta)$. Thus $W_r^I(n,\theta)$ is invariant to frequency-shifting the signal $r(n)$ by π radians/second.

B. Type-II Wigner Distribution

The type-II Wigner distribution $W_r^{II}(n,\theta)$ is defined as

$$W_r^{II}(n,\theta) = \sum_k r(n+k+1)r^*(n-k)e^{-j(2k+1)\theta}, \tag{13}$$

where n is discrete-time and θ is frequency.

For a signal $r(n)$ that is zero outside $0 \le n \le (N-1)$, the type-II Wigner distribution $W_r^{II}(n,\theta)$ is zero outside $0 \le n \le (N-2)$.

The following properties are easy to verify: $W_r^{II}(n,\theta)$ is real, $W_r^{II}(n,\theta)$ is a periodic function of θ with period 2π, $W_r^{II}(n,\theta+\pi) = -W_r^{II}(n,\theta)$, $W_{r_\pi}^{II}(n,\theta) = -W_r^{II}(n,\theta)$. Thus frequency-shifting of the signal $r(n)$ by π radians/second causes a sign change in $W_r^I(n,\theta)$.

C. Type-III Wigner Distribution

The type-III Wigner distribution $W_r^{III}(n,\theta)$ is defined in terms of the type-I and type-II Wigner distributions as follows.

$$W_r^{III}(n,\theta) = \begin{cases} W_r^I(n/2,\theta) & \text{for even} \quad n, \\ W_r^{II}((n-1)/2,\theta) & \text{for odd} \quad n. \end{cases} \tag{14}$$

Note that $W_r^{III}(n,\theta)$ is the same as the "non-aliased discrete-time Wigner distribution" derived by Chan in [7] in an effort to solve the aliasing problem of $W_r^{CM}(n,\theta)$ (c.f. (6)) defined in [5].

For a signal $r(m)$ that is zero outside $0 \le m \le (N-1)$, the type-III Wigner distribution $W_r^{III}(n,\theta)$ is zero outside $0 \le n \le 2(N-1)$. However, if we consider even n to correspond to integer values $n/2$ of time and odd n to correspond to half-integer values $n/2$ of time, then $W_r^{III}(n,\theta)$ is zero outside the time range $0 \le m \le (N-1)$.

The following properties are obvious: $W_r^{III}(n,\theta)$ is real, $W_r^{III}(n,\theta)$ is a periodic function of θ with period 2π. For even n, $W_{r_\pi}^{III}(n,\theta) = W_r^{III}(n,\theta)$. For odd n, $W_{r_\pi}^{III}(n,\theta) = -W_r^{III}(n,\theta)$.

III. A Wigner-Distribution Formulation of the ML Detection Problem for a Discrete-Time Chirp Signal

For a discrete-time signal $r(n)$ that is zero outside $0 \le n \le (N-1)$, we have shown in Appendix II that

$$\Delta_r(c_1,c_2) = \sum_{n=0}^{2N-2} W_r^{III}(n, c_1 + \frac{1}{2}c_2 n). \tag{15}$$

Thus we can calculate $\Delta_r(c_1,c_2)$ by taking the type-III Wigner distribution $W_r^{III}(n,\theta)$ of the discrete-time signal $r(n)$ and integrating it along the line with intercept c_1 (value of θ at $n=0$) and slope $\frac{1}{2}c_2$ (increment in θ per unit increment in n). This property of $W_r^{III}(n,\theta)$ was not observed in [7]. Therefore, in the context of detection of discrete-time signals, the type-III Wigner distribution $W_r^{III}(n,\theta)$ seems new.

Relationship (15) implies that we can perform ML detection by the test

$$\max_{c_1, c_2} \sum_{n=0}^{2N-2} W_r^{III}(n, c_1 + \frac{1}{2}c_2 n) \underset{\mathbf{H}_0}{\overset{\mathbf{H}_1}{\gtrless}} \gamma, \tag{16}$$

where γ, $\mathbf{H}_0$ and $\mathbf{H}_1$ are as defined for the correlator method (9).

A. Advantages of the Type-III Wigner Distribution Based Method

In Appendix III, we have derived the type-I and type-III Wigner distributions for the discrete-time chirp signal

$$s(n) = \begin{cases} e^{j(b_1 n + \frac{1}{2} b_2 n^2)} & \text{if } 0 \le n \le (N-1) \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

If the chirp signal has complex amplitude b_0 then the Wigner distributions must be scaled by $|b_0|^2$. Thus, in the absence of any noise or interference, the Wigner distributions $W_r^{I}(n,\theta)$ and $W_r^{III}(n,\theta)$ are concentrated along a straight line whose intercept of the frequency axis is b_1 and the frequency/time slope is b_2. Therefore, the visual appeal of the W_r^{CM}-based method (11) is retained by the W_r^{III}-based method (16). Moreover, any method of automatically detecting the line where W_r^{CM} is concentrated can be used for automatically detecting the line where W_r^{III} is concentrated.

As the W_r^{III}-based method (16) is mathematically equivalent to the correlator method (9), it has the same Signal-to-Noise (SNR) performance as the ML method.

The properties of the type-I and type-III Wigner distributions stated in Section II show that the range of unambiguously measurable values of b_1 can be doubled by using the W_r^{III}-based method instead of the W_r^{CM}-based method. More specifically,

- for the W_r^{CM}-based method, the interval of unambiguously measurable values of b_1 is $[-\pi/2, \pi/2]$,
- for the W_r^{III}-based method, the interval of unambiguously measurable values of b_1 is $[-\pi, \pi]$, which is the maximum possible.

IV. Conclusion

In this paper, we considered detecting a discrete-time chirp signal, in the presence of additive white Gaussian noise, via the line-integral transform of a time-frequency distribution of the observed signal. We pointed out that the popular method, in which the line-integral transform of the Wigner-Classen-Mecklenbrauker distribution is maximized, is not equivalent to the maximum-likelihood (ML) method. We derived a Wigner-type distribution with the property that maximizing its line-integral transform is equivalent to the ML method. We provided simulated Receiver Operating Curves for the Wigner-Claasen-Mecklenbrauker distribution based method and the new ML-equivalent method and demonstrated the suboptimality of the former. The use of the derived Wigner-type distribution also doubles the range of unambiguously measurable values of the initial frequency parameter b_1 of the chirp signal to the maximum possible $[-\pi, \pi]$.

Acknowledgement

The authors gratefully acknowledge the support provided by Dr. A. M. (Tony) Ponsford, Technical Director of HF radar development at Raytheon Systems Canada Ltd.

APPENDIX I

The Maximum Likelihood Method

Suppose we have observed a discrete-time signal $r(n)$, for $n = 0, \ldots, (N-1)$, and want to detect the presence or absence in $r(n)$ of a chirp signal

$$s(n) = \begin{cases} b_0 e^{j(b_1 n + \frac{1}{2} b_2 n^2)} & \text{if } 0 \le n \le (N-1), \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

with unknown parameters b_0, b_1 and b_2, with the background being additive white Gaussian noise. The classical Maximum-Likelihood (ML) method is then equivalent to the hypothesis test

$$\max_{c_0, c_1, c_2} \Lambda(c_0, c_1, c_2) \underset{\mathbf{H}_0}{\overset{\mathbf{H}_1}{\gtrless}} \eta, \tag{19}$$

where

$$\Lambda(c_0, c_1, c_2) = \sum_{n=0}^{N-1} |r(n)|^2 - \sum_{n=0}^{N-1} \left| r(n) - c_0 e^{j(c_1 n + \frac{1}{2} c_2 n^2)} \right|^2 \tag{20}$$

is the log-likelihood ratio,

- $\mathbf{H}_0$ is the Noise-Only Hypothesis $r(n) = v(n)$,
- $\mathbf{H}_1$ is the Signal-plus-Noise Hypothesis $r(n) = s(n) + v(n)$,

$v(n)$ is white Gaussian noise and η is a threshold whose value is set based on probability of error considerations. When $\mathbf{H}_1$ is considered true, the values of c_0, c_1, and c_2 that maximize $\Lambda(c_0, c_1, c_2)$ are the maximum likelihood estimates of b_0, b_1 and b_2, respectively.

By writing

$$\Lambda(c_0, c_1, c_2) = 2\Re\left(c_0^* \sum_{n=0}^{N-1} r(n) e^{-j(c_1 n + \frac{1}{2} c_2 n^2)} \right) - |c_0|^2 N, \tag{21}$$

$$= N \left(\frac{1}{N^2} \left| \sum_{n=0}^{N-1} r(n) e^{-j(c_1 n + \frac{1}{2} c_2 n^2)} \right|^2 - \left| \frac{1}{N} \sum_{n=0}^{N-1} r(n) e^{-j(c_1 n + \frac{1}{2} c_2 n^2)} - c_0 \right|^2 \right), \tag{22}$$

we conclude that for any fixed (c_1, c_2) pair, $\Lambda(c_0, c_1, c_2)$ is maximized by

$$c_0 = \frac{1}{N} \sum_{n=0}^{N-1} r(n) e^{-j(c_1 n + \frac{1}{2} c_2 n^2)}, \tag{23}$$

and for this choice of c_0, $\Lambda(c_0, c_1, c_2) = \frac{1}{N} \Delta(c_1, c_2)$, where

$$\Delta(c_1, c_2) = \left| \sum_{n=0}^{N-1} r(n) e^{-j(c_1 n + \frac{1}{2} c_2 n^2)} \right|^2. \tag{24}$$

Therefore, the ML method is equivalent to the hypothesis test

$$\max_{c_1, c_2} \Delta(c_1, c_2) \underset{\mathbf{H}_0}{\overset{\mathbf{H}_1}{\gtrless}} \gamma, \tag{25}$$

where γ is a threshold (which can be related to η). When $\mathbf{H}_1$ is considered true, the values of c_1 and c_2 that maximize $\Delta(c_1, c_2)$ are the maximum likelihood estimates of b_1 and b_2 respectively.

Thus the ML method can be implemented as a correlator in the (c_1, c_2) plane. This method is also known as the dechirp-Doppler method, i.e., first multiplying by $e^{-j\frac{1}{2}c_2 n^2}$ to obtain a pure complex exponential, or nearly so, so that there will be little or no loss due to Doppler spreading when estimating the frequency of the pure complex exponential by the conventional Doppler processing method.

APPENDIX II

Computing $\Delta_r(c_1, c_2)$ via the Wigner Distributions

For a discrete-time signal $r(n)$ that is zero outside $0 \leq n \leq (N-1)$, we show how to write $\Delta_r(c_1, c_2)$ (c.f. (8)) as the line-integral of a time-frequency distribution of $r(n)$.

We begin by writing $\Delta_r(c_1, c_2)$ as the double summation

$$\begin{aligned} \Delta_r(c_1, c_2) &= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} r(n_1) r^*(n_2) e^{-j(c_1(n_1-n_2)+\frac{1}{2}c_2(n_1^2-n_2^2))}, && (26) \\ &= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} r(n_1) r^*(n_2) e^{-j(n_1-n_2)(c_1+\frac{1}{2}c_2(n_1+n_2))}. && (27) \end{aligned}$$

Then we break the double summation into two double summations - one double summation being over n_1 and n_2 where $(n_1 - n_2)$ is even and the other double summation being over n_1 and n_2 where $(n_1 - n_2)$ is odd. Thus we define

$$\Delta_r^I(c_1, c_2) = \sum_{\substack{n_1=0 \\ (n_1-n_2) \text{ is even}}}^{N-1} \sum_{n_2=0}^{N-1} r(n_1) r^*(n_2) e^{-j(n_1-n_2)(c_1+\frac{1}{2}c_2(n_1+n_2))}, \quad (28)$$

and

$$\Delta_r^{II}(c_1, c_2) = \sum_{\substack{n_1=0 \\ (n_1-n_2) \text{ is odd}}}^{N-1} \sum_{n_2=0}^{N-1} r(n_1) r^*(n_2) e^{-j(n_1-n_2)(c_1+\frac{1}{2}c_2(n_1+n_2))}. \quad (29)$$

To compute $\Delta_r^I(c_1, c_2)$, we use the change of variables[2]

$$\begin{aligned} n_1 + n_2 &= 2m, && (30) \\ n_1 - n_2 &= 2k, && (31) \end{aligned}$$

define $l = \min(m, N-1-m)$ and obtain

$$\begin{aligned} \Delta_r^I(c_1, c_2) &= \sum_{m=0}^{N-1} \sum_{k=-l}^{l} r(m+k) r^*(m-k) e^{-j2k(c_1+c_2 m)}, && (32) \\ &= \sum_{m=0}^{N-1} W_r^I(m, c_1 + c_2 m). && (33) \end{aligned}$$

Similarly, to compute $\Delta_r^{II}(c_1, c_2)$, we use the variable change

$$n_1 + n_2 = 2m + 1, \quad (34)$$
$$n_1 - n_2 = 2k + 1, \quad (35)$$

[2]Note that $n_1 + n_2$ is even if and only if $n_1 - n_2$ is even.

define $l = \min(m, N-2-m)$ and obtain

$$\Delta_r^{II}(c_1, c_2) = \sum_{m=0}^{N-2} \sum_{k=-(l+1)}^{l} r(m+k+1)r^*(m-k)e^{-j(2k+1)(c_1+\frac{1}{2}c_2+c_2 m)}, \tag{36}$$

$$= \sum_{m=0}^{N-2} W_r^{II}(m, c_1 + \frac{1}{2}c_2 + c_2 m). \tag{37}$$

By combining (33) and (37), we obtain

$$\Delta_r(c_1, c_2) = \Delta_r^I(c_1, c_2) + \Delta_r^{II}(c_1, c_2), \tag{38}$$

$$= \sum_{m=0}^{N-1} W_r^I(m, c_1 + c_2 m) + \sum_{m=0}^{N-2} W_r^{II}(m, c_1 + \frac{1}{2}c_2 + c_2 m), \tag{39}$$

$$= \sum_{m=0}^{N-1} W_r^{III}(2m, c_1 + \frac{1}{2}c_2 2m) + \sum_{m=0}^{N-2} W_r^{III}(2m+1, c_1 + \frac{1}{2}c_2(2m+1)), \tag{40}$$

$$= \sum_{\substack{m=0 \\ \text{(m is even)}}}^{2N-2} W_r^{III}(m, c_1 + \frac{1}{2}c_2 m) + \sum_{\substack{m=0 \\ \text{(m is odd)}}}^{2N-2} W_r^{III}(m, c_1 + \frac{1}{2}c_2 m), \tag{41}$$

$$= \sum_{m=0}^{2N-2} W_r^{III}(m, c_1 + \frac{1}{2}c_2 m). \tag{42}$$

Thus we have proved (15) of Section III.

APPENDIX III
Wigner Distributions of a Discrete-Time Chirp Signal

Here we derive the type-I, type-II, and type-III Wigner distributions for the discrete-time chirp signal

$$s(n) = \begin{cases} e^{j(b_1 n + \frac{1}{2}b_2 n^2)} & \text{if } 0 \le n \le (N-1) \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

These Wigner distributions are defined in Section II.

A. Type-I Wigner Distribution

The type-I Wigner distribution $W_s^I(n, \theta)$ is defined as

$$W_s^I(n, \theta) = \sum_k s(n+k)s^*(n-k)e^{-j2k\theta}. \tag{44}$$

The signal product term $s(n+k)s^*(n-k)$ is zero outside the ranges $0 \le n \le (N-1)$ and $-\min(n, N-1-n) \le k \le \min(n, N-1-n)$. Thus $W_s^I(n, \theta) = 0$ outside $0 \le n \le (N-1)$.

Define $l = \min(n, N-1-n)$. In terms of this, for $0 \le n \le (N-1)$,

$$W_s^I(n,\theta) = \sum_{k=-l}^{l} e^{j(b_1(n+k)+\frac{1}{2}b_2(n+k)^2)} e^{-j(b_1(n-k)+\frac{1}{2}b_2(n-k)^2)} e^{-j2k\theta}, \quad (45)$$

$$= \sum_{k=-l}^{l} e^{j(b_1 2k+\frac{1}{2}b_2 4nk)} e^{-j2k\theta}, \quad (46)$$

$$= \sum_{k=-l}^{l} e^{-j(\theta-(b_1+b_2 n))2k}, \quad (47)$$

which is a sum of a geometric series that can be easily evaluated. To do this, we substitute $\alpha = \theta - (b_1 + b_2 n)$ into the summation

$$\sum_{k=-l}^{l} e^{-j2\alpha k} = \begin{cases} 2l+1 & \text{if } \alpha = 0 \bmod \pi, \\ \frac{\sin[\alpha(2l+1)]}{\sin\alpha} & \text{otherwise.} \end{cases} \quad (48)$$

Thus, for $0 \le n \le (N-1)$,

$$W_s^I(n,\theta) = \begin{cases} 2l(n)+1 & \text{if } \alpha(n) = 0 \bmod \pi, \\ \frac{\sin[\alpha(n)(2l(n)+1)]}{\sin\alpha(n)} & \text{otherwise.} \end{cases}, \quad (49)$$

where $l(n) = \min(n, N-1-n)$ and $\alpha(n) = \theta - (b_1 + b_2 n)$. In a 3-dimensional plot, $W_s^I(n,\theta)$ has ridges along the lines given by $\alpha(n) = 0 \bmod \pi$ and the common height of these ridges is $2l(n)+1$.

B. Type-II Wigner Distribution

The type-II Wigner distribution $W_s^{II}(n,\theta)$ is defined as

$$W_s^{II}(n,\theta) = \sum_k s(n+k+1)s^*(n-k)e^{-j(2k+1)\theta}. \quad (50)$$

The signal product term $s(n+k+1)s^*(n-k)$ is zero outside the ranges $0 \le n \le (N-2)$ and $-\min(n+1, N-1-n) \le k \le \min(n, N-2-n)$. Thus $W_s^{II}(n,\theta) = 0$ outside $0 \le n \le (N-2)$.

Define $l = \min(n, N-2-n)$. In terms of this, for $0 \le n \le (N-2)$,

$$W_s^{II}(n,\theta) = \sum_{k=-(l+1)}^{l} e^{j(b_1(n+k+1)+\frac{1}{2}b_2(n+k+1)^2)} e^{-j(b_1(n-k)+\frac{1}{2}b_2(n-k)^2)} e^{-j(2k+1)\theta}, \quad (51)$$

$$= \sum_{k=-(l+1)}^{l} e^{j(b_1(2k+1)+\frac{1}{2}b_2(4nk+2n+2k+1))} e^{-j(2k+1)\theta}, \quad (52)$$

$$= \sum_{k=-(l+1)}^{l} e^{-j(\theta-(b_1+\frac{1}{2}b_2+b_2 n))(2k+1)}, \quad (53)$$

$$= e^{-j(\theta-(b_1+\frac{1}{2}b_2+b_2 n))} \sum_{k=-(l+1)}^{l} e^{-j(\theta-(b_1+\frac{1}{2}b_2+b_2 n))2k}, \quad (54)$$

which is a scaled version of a sum of a geometric series that can be easily evaluated. To do this, we substitute $\alpha = \theta - (b_1 + \frac{1}{2}b_2 + b_2 n)$ into the summation

$$\sum_{k=-(l+1)}^{l} e^{-j2\alpha k} = \begin{cases} 2l+2 & \text{if } \alpha = 0 \bmod \pi, \\ e^{j\alpha}\left(\frac{\sin[2\alpha(l+1)]}{\sin\alpha}\right), \end{cases} \quad (55)$$

or directly into the scaled version

$$e^{-j\alpha}\sum_{k=-(l+1)}^{l} e^{-j2\alpha k} = \begin{cases} e^{-j\alpha}(2l+2) & \text{if } \alpha = 0 \bmod \pi, \\ \frac{\sin[2\alpha(l+1)]}{\sin\alpha}, \end{cases} \tag{56}$$

$$= \begin{cases} (2l+2) & \text{if } \alpha = 0 \bmod 2\pi, \\ -(2l+2) & \text{if } \alpha = \pi \bmod 2\pi, \\ \frac{\sin[2\alpha(l+1)]}{\sin\alpha}. \end{cases} \tag{57}$$

Thus, for $0 \le n \le (N-2)$,

$$W_s^{II}(n,\theta) = \begin{cases} 2l(n)+2 & \text{if } \alpha(n) = 0 \bmod 2\pi, \\ -(2l(n)+2) & \text{if } \alpha(n) = \pi \bmod 2\pi, \\ \frac{\sin[2\alpha(n)(l(n)+1)]}{\sin\alpha(n)} & \text{otherwise.} \end{cases}, \tag{58}$$

where $l(n) = \min(n, N-2-n)$ and $\alpha(n) = \theta - (b_1 + \frac{1}{2}b_2 + b_2 n)$. In a 3-dimensional plot, $W_s^{II}(n,\theta)$ has ridges along the lines given by $\alpha(n) = 0 \bmod 2\pi$ and the common height of these ridges is $2l(n)+2$. In a 3-dimensional plot, $W_s^{II}(n,\theta)$ also has valleys along the lines given by $\alpha(n) = \pi \bmod 2\pi$ and the common depth of these ridges is $2l(n)+2$.

C. Type-III Wigner Distribution

The type-III Wigner distribution $W_s^{III}(n,\theta)$ is defined in terms of the type-I and type-II Wigner distributions as

$$W_s^{III}(n,\theta) = \begin{cases} W_s^{I}(n/2,\theta) & \text{for even } n, \\ W_s^{II}((n-1)/2,\theta) & \text{for odd } n. \end{cases} \tag{59}$$

Define $l(n) = \min(\frac{n}{2}, N-1-\frac{n}{2})$ and $\alpha(n) = \theta - (b_1 + \frac{1}{2}b_2 n)$.
For n even and $0 \le n \le 2(N-1)$,

$$W_s^{III}(n,\theta) = \begin{cases} 2l(n)+1 & \text{if } \alpha(n) = 0 \bmod \pi, \\ \frac{\sin[\alpha(n)(2l(n)+1)]}{\sin\alpha(n)} & \text{otherwise.} \end{cases} \tag{60}$$

For n odd and $0 \le n \le 2(N-1)$,

$$W_s^{III}(n,\theta) = \begin{cases} 2l(n)+1 & \text{if } \alpha(n) = 0 \bmod 2\pi, \\ -[2l(n)+1] & \text{if } \alpha(n) = \pi \bmod 2\pi, \\ \frac{\sin[\alpha(n)(2l(n)+1)]}{\sin\alpha(n)} & \text{otherwise.} \end{cases} \tag{61}$$

Combining the above, for all $0 \le n \le 2(N-1)$,

$$W_s^{III}(n,\theta) = \begin{cases} 2l(n)+1 & \text{if } \alpha(n) = 0 \bmod 2\pi, \\ (-1)^n[2l(n)+1] & \text{if } \alpha(n) = \pi \bmod 2\pi, \\ \frac{\sin[\alpha(n)(2l(n)+1)]}{\sin\alpha(n)} & \text{otherwise.} \end{cases} \tag{62}$$

In a 3-dimensional plot, $W_s^{III}(n,\theta)$ has ridges along the lines given by $\alpha(n) = 0 \bmod 2\pi$ and the common height of these ridges is $2l(n)+1$. In a 3-dimensional plot, $W_s^{III}(n,\theta)$ also has oscillations along the lines given by $\alpha(n) = \pi \bmod 2\pi$; the common period of these oscillations is one and the common (instantaneous) amplitude is $2l(n)+1$.

APPENDIX IV

Receiver Operating Curve Comparison

Here we present the Receiver Operating Curves (ROCs) obtained by simulation of the Type-I and Type-III Wigner distribution based methods for the case $N = 128$ and Output SNR $= 7$ dB, where Output SNR is defined as $N\left(\frac{a}{\sigma}\right)^2$. For simplicity, the parameters of the chirp signal were assumed to be known. The suboptimality of the Type-I Wigner distribution based method can be clearly seen.

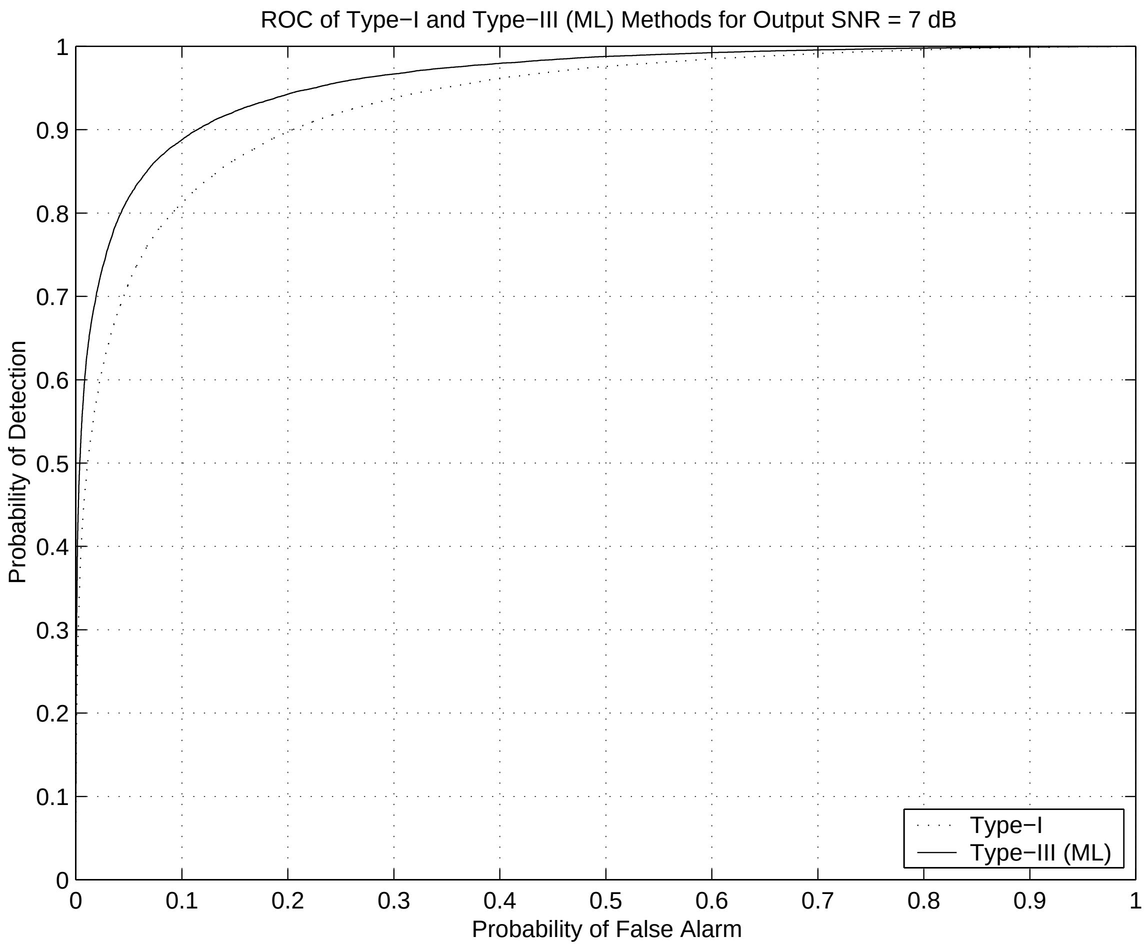

Figure 1: ROCs for probability of false alarm ranging from 0 to 1

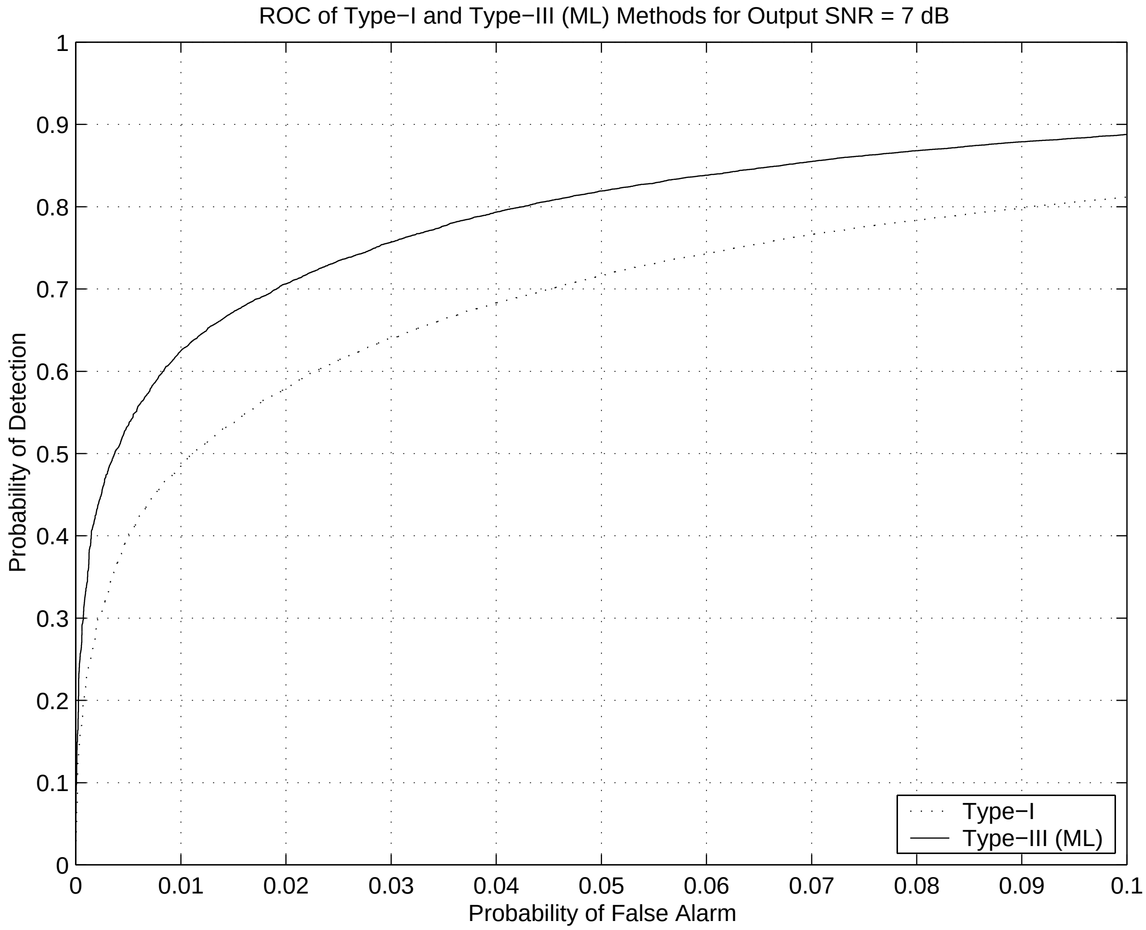

Figure 2: ROCs for probability of false alarm ranging from 0 to 0.1

REFERENCES

[1] T. A. C. M. Claasen and W. F. G. Mecklenbrauker, "The Wigner Distribution - A Tool for Time-Frequency Signal Analysis, Part-I: Continuous-Time Signals", Philips Journal of Research, Vol. 35, No. 3, 1980, pp. 217-250.

[2] S. Kay and G. F. Boudreaux-Bartels, "On the Optimality of the Wigner Distribution for Detection", Proc. ICASSP '85, pp. 27.2.1-4.

[3] W. Li, "Wigner Distribution Method Equivalent to Dechirp Method for Detecting a Chirp Signal", IEEE Trans. Acoustics, Speech, and Signal Processing, Vol. ASSP-35, No. 8, August 1987, pp. 1210-1211.

[4] P. Flandrin, "A Time-Frequency Formulation of Optimum Detection", IEEE Trans. Acoustics, Speech, and Signal Processing, Vol. 36, No. 9, September 1988, pp. 1377-1384.

[5] T. A. C. M. Claasen and W. F. G. Mecklenbrauker, "The Wigner Distribution - A Tool for Time-Frequency Signal Analysis, Part-II: Discrete-Time Signals", Philips Journal of Research, Vol. 35, Nos. 4/5, 1980, pp. 276-300.

[6] S. Barbarossa, "Analysis of Multicomponent LFM Signals by a Combined Wigner-Hough Transform", IEEE Trans. Signal Processing, Vol. 43, No. 6, June 1995, pp. 1511-1515.

[7] D. S. K. Chan, "A Non-Aliased Discrete-Time Wigner Distribution for Time-Frequency Signal Analysis", Proc. ICASSP '82, pp. 1333-1336.

[8] A.Yasotharan, T. Thayaparan, Strengths and limitations of the Fourier method for detecting accelerating targets by pulse Doppler radar, IEE Proceedings - Radar, Sonar, Navigation, Vol. 149, No. 2, April 2002, pp. 83 - 88.

[9] P. Flandrin, "Chirps everywhere," Invited talk given at the CEMRACS "Mthodes Mathmatiques en Traitement d'Images", Marseille, August 2002. Available in http://perso.ens-lyon.fr/patrick.flandrin/publirecentes.html.